Written test - Mathematics

2 hours

The use of documents and calculators is forbidden.

The quality of writing is an important element for the evaluation.

Exercise 1

Q. 1.1 Justify the existence of

\[ I = \int_{\mathbb{R}} e^{-x^2} \, dx. \]

Q. 1.2 Compute the value of the integral \( I \).

Exercise 2

Let \( P \) be a polynomial with real coefficients, \( P \in \mathbb{R}[X] \). Assume all its roots are real. Let \( \alpha \in \mathbb{R} \). We want to prove that all the roots of \( P' + \alpha P \) are real too.

Q. 2.1 Assume at first that \( P \) has only simple roots. Show that \( P' + \alpha P \), where \( P' \) the derivative polynomial of \( P \), has only real simple roots, as well.

Q. 2.2 We do not assume that the roots of \( P \) are all simple anylonger. Show that the roots of \( P' + \alpha P \) are all real.
Exercise 3

Q. 3.1 Determine the convergence radius R of the series \[ \sum \frac{n^n}{n!} z^n. \]

Q. 3.2 Study the convergence of the series on the circle of center 0 and radius R.

Exercise 4

Q. 4.1 Study the linear independence of the following families:

1. \( f_n : x \mapsto (\sin(x))^n, \ n \in \mathbb{N} \)
2. \( f_a : x \mapsto |x - a|, \ a \in \mathbb{R} \)
3. \( f_\alpha : x \mapsto \exp(\alpha x), \ \alpha \in \mathbb{C} \)
4. \( (x \mapsto (\sin(x))^n), (x \mapsto (\cos(x))^n), \ n \in \mathbb{N}. \)

Q. 4.2 Let \( n \geq 1. \) Let \( f_i : \mathbb{C} \rightarrow \mathbb{C}, \ i \in \{1, \ldots, n\} \) be a family of functions. Set
\[
A : \mathbb{C}^n \rightarrow \mathcal{M}_1(\mathbb{C}) \quad \text{where} \quad (x_1, \ldots, x_n) \mapsto (f_i(x_j))_{1 \leq i, j \leq n}
\]
Find a necessary and sufficient condition on the function \( \det(A) \) so that the family \((f_i)_{1 \leq i \leq n}\) is linearly independent.

Exercise 5

Let \( X \) be a real random variable following Poisson's law with parameter \( \lambda. \) Let \( Y \) be defined as
\[
Y = \begin{cases} 
\frac{X}{2} & \text{if } X \text{ is even} \\
0 & \text{if } X \text{ is odd}
\end{cases}
\]

Q. 5.1 Determine the law of \( Y. \)

Q. 5.2 Compute the expectation of \( Y. \)

Q. 5.3 Compute the variance of \( Y. \)